

# A New Approach Towards a Conjecture on Intersecting Three Longest Paths

Shinya Fujita<sup>1</sup>, Michitaka Furuya<sup>2</sup>, Reza Naserasr<sup>3</sup>, Kenta Ozeki<sup>4</sup>

## Abstract

In 1966, T. Gallai asked whether every connected graph has a vertex that appears in all longest paths. Since then this question has attracted much attention and many work has been done in this topic. One important open question in this area is to ask whether any three longest paths contains a common vertex in a connected graph. It was conjectured that the answer to this question is positive. In this paper, we propose a new approach in view of distances among longest paths in a connected graph, and give a substantial progress towards the conjecture along the idea.

## 1 Introduction

In [4] Gallai asked whether every connected graph has a vertex that appears in all longest paths. This question has attracted much attention and many work has been done around this area of study. The answer to this question is false as stated; actually several counterexamples were given in [8, 9, 10]. A graph  $G$  is *hypotraceable* if  $G$  has no Hamiltonian path but every vertex-deleted subgraph  $G - v$  has. Note that hypotraceable graphs constitute a large class of counterexamples. Thomassen [7] showed that there exist infinitely many planar hypotraceable graphs, meaning that there exist infinitely many counterexamples towards the question.

Yet there are classes of graphs for which the answer to Gallai's question is positive. To see this, note that, in a tree, all longest paths must contain its center(s). Klavžar and Petkovšek [6] showed that the answer is also positive for split graphs, cacti, and some other classes of graphs. Balister et al. [2] obtained a similar result for the class of circular arc graphs.

Regarding Gallai's question, what happens if we consider the intersection of a smaller number of longest paths? While we can easily check that every two longest

---

<sup>1</sup>International College of Arts and Sciences, Yokohama City University, 22-2 Seto, Kanazawa-ku, Yokohama 236-0027, Japan; shinya.fujita.ph.d@gmail.com

<sup>2</sup>Department of Mathematical Information Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan; michitaka.furuya@gmail.com

<sup>3</sup>CNRS, LRI, UMR8623, Univ. Paris-Sud 11, F-91405 Orsay Cedex, France; reza@lri.fr

<sup>4</sup>National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-Ku, Tokyo 101-8430, Japan and JST, ERATO, Kawarabayashi Large Graph Project, Japan; ozeki@nii.ac.jp

paths share a vertex, it is not known whether every three longest paths also share a vertex. In [5] it appears as a conjecture, which has originally been asked by Zamfirescu since the 1980s (see [11]).

**Conjecture 1** *For every connected graph, any three of its longest paths have a common vertex.*

So far, very little progress has been made on this conjecture. Axenovich [1] proved that Conjecture 1 is true for connected outerplanar graphs, and de Rezende et al. [3] proved that Conjecture 1 is true for connected graphs in which all nontrivial blocks are Hamiltonian.

In this paper, we introduce a new graph parameter in view of distances among longest paths in a connected graph. To state this, we give some basic definitions. For a graph  $G$ , let  $P$  be a path in  $G$ , and let  $x$  and  $y$  be the end-vertices of  $P$ . Note that  $|V(P)| = 1$  if and only if  $x = y$ . For  $X, Y \subseteq V(G)$ ,  $P$  is called an  $X$ - $Y$  path if  $V(P) \cap X = \{x\}$  and  $V(P) \cap Y = \{y\}$ . Let  $u, v \in V(P)$ . We let  $uPv$  denote the  $\{u\}$ - $\{v\}$  path on  $P$ . Furthermore, we let  $\check{u}Pv = uPv - u$ ,  $uP\check{v} = uPv - v$  and  $\check{u}P\check{v} = uPv - \{u, v\}$ .

Let  $G$  be a connected graph. Let  $l(G)$  be the length of any longest path in  $G$ , and let  $\mathcal{L}(G)$  be the set of longest paths of  $G$ ; thus  $\mathcal{L}(G) = \{P \mid P \text{ is a path in } G \text{ with } |V(P)| = l(G) + 1\}$ . For  $x, y \in V(G)$  let  $d_G(x, y)$  be the distance between  $x$  and  $y$  in  $G$  (i.e., the length of a shortest path joining  $x$  and  $y$  in  $G$ ). Also for a vertex  $x \in V(G)$  and a subset  $U \subseteq V(G)$ , let  $d_G(x, U) = \min\{d_G(x, y) \mid y \in U\}$ . For  $\mathcal{P} \subseteq \mathcal{L}(G)$ , let  $f(G, \mathcal{P}) = \min\{\sum_{P \in \mathcal{P}} d_G(v, V(P)) \mid v \in V(G)\}$ .

Using this graph parameter, we can formulate Conjecture 1 as follows.

**Conjecture 2** *Let  $G$  be a connected graph, and let  $\mathcal{P}$  be a subset of  $\mathcal{L}(G)$  with  $|\mathcal{P}| = 3$ . Then  $f(G, \mathcal{P}) = 0$ .*

As mentioned before, it is easy to check that any two longest paths of a connected graph have a common vertex. We now give the proof in this context.

**Proposition 3** *Let  $G$  be a connected graph, and let  $\mathcal{P}$  be a subset of  $\mathcal{L}(G)$  with  $|\mathcal{P}| = 2$ . Then  $f(G, \mathcal{P}) = 0$ .*

*Proof.* Write  $\mathcal{P} = \{P_1, P_2\}$ , and for each  $i \in \{1, 2\}$ , let  $u_i$  and  $v_i$  be the end-vertices of  $P_i$ . Since  $G$  is connected,  $G$  has a  $V(P_1)$ - $V(P_2)$  path  $Q$ . Note that  $V(P_1) \cap V(P_2) \neq \emptyset$  if and only if  $|V(Q)| = 1$ . For each  $i \in \{1, 2\}$ , write  $V(P_i) \cap V(Q) = \{w_i\}$ . We may assume that  $|V(u_i P_i w_i)| \geq |V(v_i P_i w_i)|$  for each  $i \in \{1, 2\}$ . Then the length of the path  $u_1 P_1 w_1 Q w_2 P_2 u_2$  in  $G$  is  $(|V(u_1 P_1 w_1)| - 1) + (|V(Q)| - 1) + (|V(u_2 P_2 w_2)| - 1)$ . On the other hand, for each  $i \in \{1, 2\}$ ,  $|V(u_i P_i w_i)| - 1 \geq ((|V(u_i P_i w_i)| - 1) + (|V(v_i P_i w_i)| - 1))/2 = (|V(P_i)| - 1)/2 = l(G)/2$ . Consequently,

$$\begin{aligned} & (|V(u_1 P_1 w_1)| - 1) + (|V(u_2 P_2 w_2)| - 1) \\ & \geq \frac{l(G)}{2} + \frac{l(G)}{2} \\ & = l(G) \\ & \geq (|V(u_1 P_1 w_1)| - 1) + (|V(Q)| - 1) + (|V(u_2 P_2 w_2)| - 1). \end{aligned}$$

This leads to  $|V(Q)| = 1$ , and hence  $V(P_1) \cap V(P_2) \neq \emptyset$ .  $\square$

In this paper, we give an upper bound of  $f(G, \mathcal{P})$  with  $|\mathcal{P}| = 3$ , which is linear in terms of  $|V(G)|$ .

**Theorem 4** *Let  $G$  be a connected graph of order  $n$ , and let  $\mathcal{P}$  be a subset of  $\mathcal{L}(G)$  with  $|\mathcal{P}| = 3$ . Then  $f(G, \mathcal{P}) \leq (n + 6)/13$ .*

After proving this bound in Section 2, in the follow-up section we show that to prove the conjecture it would be enough to improve our linear bound to any non-decreasing sublinear bound. Namely, we propose an equivalent conjecture towards Conjecture 1 in terms of the function  $f(G, \mathcal{P})$ .

## 2 Proof of Theorem 4

We start with some lemmas.

For a set  $\mathcal{P}$  of graphs and  $P \in \mathcal{P}$ , set  $X_{\mathcal{P}}(P) = V(P) - (\bigcup_{P' \in \mathcal{P} - \{P\}} V(P'))$ .

**Lemma 5** *Let  $G$  be a connected graph of order  $n$ , and let  $\mathcal{P} \subseteq \mathcal{L}(G)$  with  $|\mathcal{P}| = 3$ . If  $f(G, \mathcal{P}) > 0$ , then  $n \geq (3l(G) + \sum_{P \in \mathcal{P}} |X_{\mathcal{P}}(P)| + 3)/2$ .*

*Proof.* Write  $\mathcal{P} = \{P_1, P_2, P_3\}$ . Since  $\bigcap_{1 \leq i \leq 3} V(P_i) = \emptyset$ ,

$$n \geq \left| \bigcup_{1 \leq i \leq 3} V(P_i) \right| = \sum_{1 \leq i \leq 3} |X_{\mathcal{P}}(P_i)| + \sum_{1 \leq i < j \leq 3} |V(P_i) \cap V(P_j)|. \quad (2.1)$$

Since  $l(G) + 1 = |V(P_i)| = |X_{\mathcal{P}}(P_i)| + \sum_{j \neq i} |V(P_i) \cap V(P_j)|$  for each  $1 \leq i \leq 3$ ,

$$\begin{aligned} 3l(G) + 3 &= \sum_{1 \leq i \leq 3} |X_{\mathcal{P}}(P_i)| + \sum_{1 \leq i \leq 3} \left( \sum_{j \neq i} |V(P_i) \cap V(P_j)| \right) \\ &= \sum_{1 \leq i \leq 3} |X_{\mathcal{P}}(P_i)| + 2 \sum_{1 \leq i < j \leq 3} |V(P_i) \cap V(P_j)|. \end{aligned} \quad (2.2)$$

By (2.1) and (2.2),

$$\begin{aligned} n &\geq \sum_{1 \leq i \leq 3} |X_{\mathcal{P}}(P_i)| + \sum_{1 \leq i < j \leq 3} |V(P_i) \cap V(P_j)| \\ &= \sum_{1 \leq i \leq 3} |X_{\mathcal{P}}(P_i)| + (3l(G) + 3 - \sum_{1 \leq i \leq 3} |X_{\mathcal{P}}(P_i)|)/2 \\ &= (3l(G) + 3 + \sum_{1 \leq i \leq 3} |X_{\mathcal{P}}(P_i)|)/2. \end{aligned}$$

Thus we get the desired conclusion.  $\square$

For a set  $\mathcal{P}$  of three paths and  $P \in \mathcal{P}$ , let  $t_{\mathcal{P}}(P)$  be the number of  $V(P_1)$ - $V(P_2)$  paths on  $P$ , where  $\mathcal{P} - \{P\} = \{P_1, P_2\}$ . If  $\mathcal{P}$  consists of three longest paths of a connected graph, then  $t_{\mathcal{P}}(P) \geq 1$  for every  $P \in \mathcal{P}$  by Proposition 3.

**Lemma 6** Let  $G$  be a connected graph, and let  $\mathcal{P} \subseteq \mathcal{L}(G)$  with  $|\mathcal{P}| = 3$ . Then  $|X_{\mathcal{P}}(P)| \geq t_{\mathcal{P}}(P)(f(G, \mathcal{P}) - 1)$  for each  $P \in \mathcal{P}$ .

*Proof.* We may assume that  $f(G, \mathcal{P}) \geq 1$ . Write  $\mathcal{P} - \{P\} = \{P_1, P_2\}$ , and let  $\mathcal{Q}$  be the set of  $V(P_1)$ - $V(P_2)$  paths on  $P$ . Note that every path in  $\mathcal{Q}$  has order at least two and  $|\mathcal{Q}| = t_{\mathcal{P}}(P)$ . Let  $Q \in \mathcal{Q}$ , and let  $u$  and  $v$  be the end-vertices of  $Q$  with  $u \in V(P_1)$  and  $v \in V(P_2)$ . Then  $V(Q) \cap X_{\mathcal{P}}(P) = V(Q) - \{u, v\}$ . Since  $u \in V(P) \cap V(P_1)$ ,  $f(G, \mathcal{P}) \leq \sum_{P' \in \mathcal{P}} d_G(u, V(P')) = d_G(u, V(P_2)) \leq d_G(u, v) \leq |V(Q)| - 1$ . Hence  $|V(Q) \cap X_{\mathcal{P}}(P)| = |V(Q)| - 2 \geq f(G, \mathcal{P}) - 1$ . Since  $Q$  is arbitrary,

$$\sum_{Q \in \mathcal{Q}} |V(Q) \cap X_{\mathcal{P}}(P)| \geq t_{\mathcal{P}}(P)(f(G, \mathcal{P}) - 1). \quad (2.3)$$

Clearly, each vertex in  $X_{\mathcal{P}}(P)$  belongs to at most one path in  $\mathcal{Q}$ . This together with (2.3) implies that  $|X_{\mathcal{P}}(P)| \geq |\bigcup_{Q \in \mathcal{Q}} (V(Q) \cap X_{\mathcal{P}}(P))| = \sum_{Q \in \mathcal{Q}} |V(Q) \cap X_{\mathcal{P}}(P)| \geq t_{\mathcal{P}}(P)(f(G, \mathcal{P}) - 1)$ .  $\square$

**Lemma 7** Let  $G$  be a connected graph, and let  $\mathcal{P} \subseteq \mathcal{L}(G)$  with  $|\mathcal{P}| = 3$ . If there exists a path  $P \in \mathcal{P}$  with  $t_{\mathcal{P}}(P) = 1$ , then  $f(G, \mathcal{P}) = 0$ .

*Proof.* Suppose that  $f(G, \mathcal{P}) > 0$ . Let  $u$  and  $v$  be the end-vertices of  $P$ . Write  $\mathcal{P} - \{P\} = \{P_1, P_2\}$ , and for each  $i \in \{1, 2\}$ , let  $w_i$  be the vertex which is contained in  $P_i$  and the unique  $V(P_1)$ - $V(P_2)$  path on  $P$  (see Figure 1). We may assume that  $|V(uPw_1)| \leq |V(uPw_2)|$ . Since  $f(G, \mathcal{P}) > 0$ ,  $w_1 \neq w_2$ , and hence  $|V(w_1Pv)| > |V(w_2Pv)|$ . Furthermore, we may assume that  $|V(uPw_1)| \leq |V(vPw_2)|$ . Since  $l(G) = |V(uPw_1)| + |V(w_1Pv)| - 2$ ,

$$\begin{aligned} |V(w_1Pv)| &> \frac{|V(w_1Pv)|}{2} + \frac{|V(w_2Pv)|}{2} \\ &= \frac{l(G) - |V(uPw_1)| + 2}{2} + \frac{|V(w_2Pv)|}{2} \\ &\geq \frac{l(G) - |V(vPw_2)| + 2}{2} + \frac{|V(w_2Pv)|}{2} \\ &= \frac{l(G) + 2}{2}. \end{aligned} \quad (2.4)$$

Let  $u_1$  and  $v_1$  be the end-vertices of  $P_1$ . We may assume that  $|V(u_1P_1w_1)| \geq |V(w_1P_1v_1)|$ . Since  $l(G) = |V(u_1P_1w_1)| + |V(w_1P_1v_1)| - 2$ ,

$$|V(u_1P_1w_1)| \geq \frac{|V(u_1P_1w_1)| + |V(w_1P_1v_1)|}{2} = \frac{l(G) + 2}{2}. \quad (2.5)$$

By (2.4) and (2.5),  $|V(u_1P_1w_1)| + |V(w_1Pv)| - 2 > (l(G) + 2)/2 + (l(G) + 2)/2 - 2 = l(G)$ . By the assumption that  $t_{\mathcal{P}}(P) = 1$ , the path  $\tilde{w}_1Pv$  contains no vertex in  $V(P_1)$ . Hence  $P_1^{(1)} = u_1P_1w_1Pv$  is a path in  $G$  with length  $|V(u_1P_1w_1)| + |V(w_1Pv)| - 2 > l(G)$ , which is a contradiction.  $\square$

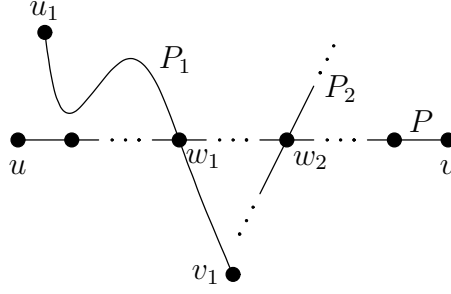


Figure 1: paths in  $\mathcal{P}$

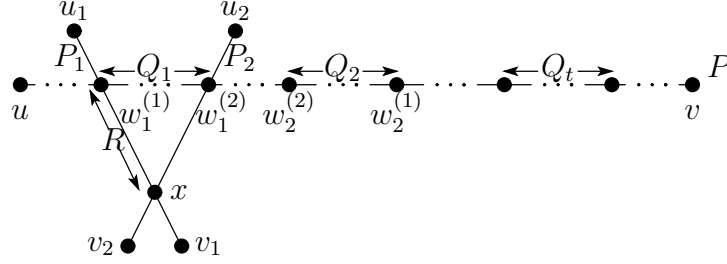


Figure 2: paths in  $\mathcal{P}$

*Proof of Theorem 4.* We may assume that  $f(G, \mathcal{P}) \geq 1$ . Choose  $P \in \mathcal{P}$  so that  $t = t_{\mathcal{P}}(P)$  is as small as possible. Then  $t_{\mathcal{P}}(P) \geq 2$  by Lemma 7. Let  $u$  and  $v$  be the end-vertices of  $P$ . Write  $\mathcal{P} - \{P\} = \{P_1, P_2\}$ , and let  $u_i$  and  $v_i$  be the end-vertices of  $P_i$  for each  $i \in \{1, 2\}$ . Let  $Q_1, Q_2, \dots, Q_t$  be the  $V(P_1)$ - $V(P_2)$  paths on  $P$  which are aligned on  $P$  in order of indices with initial point  $u$  (i.e. for each  $2 \leq i \leq t$ , the unique  $\{u\}$ - $V(Q_i)$  path on  $P$  contains  $\bigcup_{1 \leq j \leq i-1} V(Q_j)$ ). We may assume that the length of the unique  $\{u\}$ - $V(Q_1)$  path on  $P$  is at least that of the unique  $\{v\}$ - $V(Q_t)$  path on  $P$ . For each  $1 \leq i \leq t$  and each  $j \in \{1, 2\}$ , write  $V(Q_i) \cap V(P_j) = \{w_i^{(j)}\}$ . We may assume that  $|V(uPw_1^{(1)})| \leq |V(uPw_1^{(2)})|$ . Let  $R$  be a  $\{w_1^{(1)}\}$ - $V(P_2)$  path on  $P_1$ , and write  $V(R) \cap V(P_2) = \{x\}$ . For each  $i \in \{1, 2\}$ , we may assume that  $|V(u_iP_iw_1^{(i)})| \leq |V(u_iP_ix)|$  (see Figure 2).

Since  $w_1^{(1)} \in V(P) \cap V(P_1)$ ,  $f(G, \mathcal{P}) \leq \sum_{P' \in \mathcal{P}} d_G(w_1^{(1)}, V(P')) = d_G(w_1^{(1)}, V(P_2)) \leq \min\{d_G(w_1^{(1)}, w_1^{(2)}), d_G(w_1^{(1)}, x)\} \leq \min\{|V(Q_1)| - 1, |V(R)| - 1\}$ . Hence

$$|V(Q_1)| \geq f(G, \mathcal{P}) + 1 \quad \text{and} \quad |V(R)| \geq f(G, \mathcal{P}) + 1. \quad (2.6)$$

Since  $w_1^{(2)}Q_1\check{w}_1^{(1)}$  contains no vertex in  $V(P_1)$ ,  $w_1^{(2)}Q_1w_1^{(1)}Rx$  is a path in  $G$ . Furthermore, since  $\check{w}_1^{(2)}Q_1w_1^{(1)}P_1\check{x}$  contains no vertex in  $V(P_2)$ ,

- (i)  $S_1 = v_2P_2w_1^{(2)}Q_1w_1^{(1)}R\check{x}$ ,
  - (ii)  $S_2 = u_2P_2w_1^{(2)}Q_1w_1^{(1)}RxP_2v_2$  and
  - (iii)  $S_3 = u_2P_2xRw_1^{(1)}Q_1\check{w}_1^{(2)}$ .
- are paths in  $G$  (see Figure 3).

Since the length of  $S_1$  is  $(|V(v_2P_2w_1^{(2)}| - 1) + (|V(Q_1)| - 1) + (|V(w_1^{(1)}R\check{x})| - 1)$  and  $|V(w_1^{(1)}R\check{x})| = |V(R)| - 1$ , we have  $(|V(v_2P_2w_1^{(2)}| - 1) + (|V(w_1^{(2)}P_2u_2)| - 1) =$

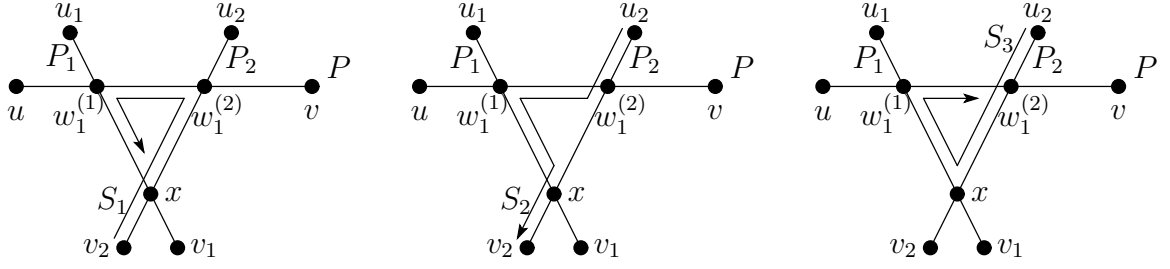


Figure 3: path  $S_i$

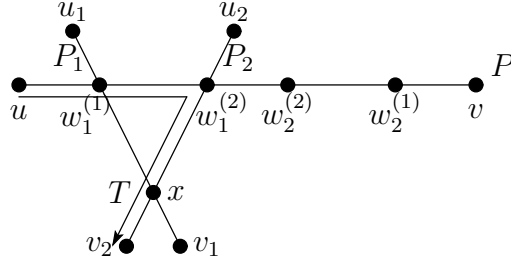


Figure 4: path  $T$

$|V(P_2)| - 1 = l(G) \geq (|V(v_2 P_2 w_1^{(2)}| - 1) + (|V(Q_1)| - 1) + (|V(R)| - 2)$ . This together with (2.6) leads to

$$|V(u_2 P_2 w_1^{(2)})| \geq |V(Q_1)| + |V(R)| - 2 \geq 2f(G, \mathcal{P}). \quad (2.7)$$

By comparing the length of  $P_2$  and  $S_2$  and (2.6), we have

$$|V(w_1^{(2)} P_2 x)| \geq |V(Q_1)| + |V(R)| - 1 \geq 2f(G, \mathcal{P}) + 1. \quad (2.8)$$

By comparing the length of  $P_2$  and  $S_3$  and (2.6), we also have

$$|V(x P_2 v_2)| \geq |V(Q_1)| + |V(R)| - 2 \geq 2f(G, \mathcal{P}). \quad (2.9)$$

Therefore

$$\begin{aligned} l(G) &= |V(P_2)| - 1 \\ &= |V(u_2 P_2 w_1^{(2)})| + |V(w_1^{(2)} P_2 x)| + |V(x P_2 v_2)| - 3 \\ &\geq 2f(G, \mathcal{P}) + (2f(G, \mathcal{P}) + 1) + 2f(G, \mathcal{P}) - 3 \\ &= 6f(G, \mathcal{P}) - 2. \end{aligned} \quad (2.10)$$

**Case 1:**  $t_{\mathcal{P}}(P) = 2$ .

Note that  $|V(v P w_2^{(1)})| \leq |V(v P w_2^{(2)})|$ . Since the path  $u P w_1^{(2)}$  contains no vertex in  $V(P_2)$ ,  $T = u P w_1^{(2)} P_2 v_2$  is a path in  $G$  (see Figure 4). Since the length of  $T$  is  $(|V(u P w_1^{(1)})| - 1) + (|V(Q_1)| - 1) + (|V(w_1^{(2)} P_2 v_2)| - 1)$ ,  $(|V(u_2 P_2 w_1^{(2)})| - 1) +$

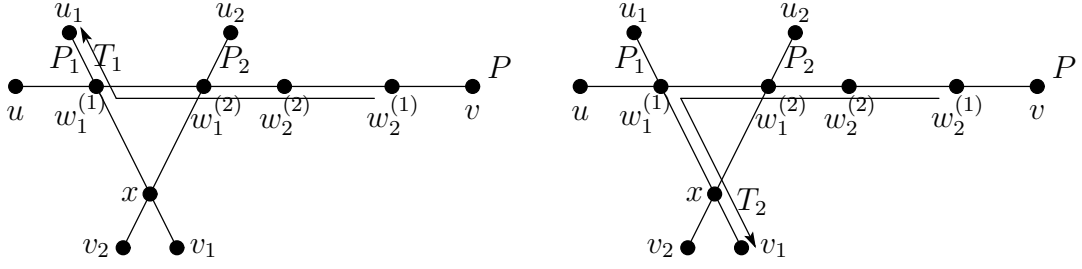


Figure 5: path  $T_i$

$(|V(w_1^{(2)}P_2v_2)| - 1) = |V(P_2)| - 1 = l(G) \geq (|V(uPw_1^{(1)})| - 1) + (|V(Q_1)| - 1) + (|V(w_1^{(2)}P_2v_2)| - 1)$ . This together with (2.6) leads to

$$|V(u_2P_2w_1^{(2)})| \geq |V(uPw_1^{(1)})| + |V(Q_1)| - 1 \geq |V(uPw_1^{(1)})| + f(G, \mathcal{P}). \quad (2.11)$$

Since the path  $\check{w}_2^{(1)}P\check{w}_1^{(1)}$  contains no vertex in  $V(P_1)$ , both  $T_1 = \check{w}_2^{(1)}Pw_1^{(1)}P_1u_1$  and  $T_2 = \check{w}_2^{(1)}Pw_1^{(1)}P_1v_1$  are paths in  $G$  (see Figure 5). Since the length of  $T_1$  is  $(|V(\check{w}_2^{(1)}Pw_1^{(1)})| - 1) + (|V(w_1^{(1)}P_1u_1)| - 1)$ , we have  $(|V(vPw_2^{(1)})| - 1) + (|V(w_2^{(1)}Pw_1^{(1)})| - 1) + (|V(w_1^{(1)}Pu)| - 1) = |V(P)| - 1 = l(G) \geq (|V(\check{w}_2^{(1)}Pw_1^{(1)})| - 1) + (|V(w_1^{(1)}P_1u_1)| - 1)$ . Consequently, we have  $|V(vPw_2^{(1)})| + |V(w_1^{(1)}Pu)| \geq |V(w_1^{(1)}P_1u_1)|$ . By comparing the length of  $P$  and  $T_2$ , we also have  $|V(vPw_2^{(1)})| + |V(w_1^{(1)}Pu)| \geq |V(w_1^{(1)}P_1v_1)|$ . Hence

$$\begin{aligned} l(G) &= |V(P_1)| - 1 \\ &= |V(u_1P_1w_1^{(1)})| + |V(w_1^{(1)}P_1v_1)| - 2 \\ &\leq 2(|V(vPw_2^{(1)})| + |V(w_1^{(1)}Pu)|) - 2. \end{aligned} \quad (2.12)$$

Recall that the length of the unique  $\{u\}$ - $V(Q_1)$  path on  $P$  (i.e.  $uPw_1^{(1)}$ ) is at least that of the unique  $\{v\}$ - $V(Q_2)$  path on  $P$  (i.e.  $vPw_2^{(1)}$ ). Hence  $|V(uPw_1^{(1)})| \geq |V(vPw_2^{(1)})|$ . By (2.12),  $l(G) \leq 2(|V(vPw_2^{(1)})| + |V(w_1^{(1)}Pu)|) - 2 \leq 4|V(uPw_1^{(1)})| - 2$ , and so  $|V(uPw_1^{(1)})| \geq (l(G) + 2)/4$ . This together with (2.11) implies that

$$|V(u_2P_2w_1^{(2)})| \geq \frac{l(G) + 2}{4} + f(G, \mathcal{P}). \quad (2.13)$$

By (2.8), (2.9) and (2.13),

$$\begin{aligned} l(G) &= |V(P_2)| - 1 \\ &= |V(u_2P_2w_1^{(2)})| + |V(w_1^{(2)}P_2x)| + |V(xP_2v_2)| - 3 \\ &\geq \left(\frac{l(G) + 2}{4} + f(G, \mathcal{P})\right) + (2f(G, \mathcal{P}) + 1) + 2f(G, \mathcal{P}) - 3 \\ &= \frac{l(G) - 6}{4} + 5f(G, \mathcal{P}), \end{aligned}$$

and so

$$l(G) \geq \frac{20f(G, \mathcal{P}) - 6}{3}. \quad (2.14)$$

By the choice of  $P$ ,  $t_{\mathcal{P}}(P') \geq 2$  for every  $P' \in \mathcal{P}$ . By Lemma 6,  $\sum_{P' \in \mathcal{P}} |X_{\mathcal{P}}(P')| \geq \sum_{P' \in \mathcal{P}} t_{\mathcal{P}}(P')(f(G, \mathcal{P}) - 1) \geq 6(f(G, \mathcal{P}) - 1)$ . This together with Lemma 5 and (2.14) implies that

$$\begin{aligned} n &\geq \frac{3l(G) + \sum_{P' \in \mathcal{P}} |X_{\mathcal{P}}(P')| + 3}{2} \\ &\geq \frac{3 \cdot \frac{20f(G, \mathcal{P}) - 6}{3} + 6(f(G, \mathcal{P}) - 1) + 3}{2} \\ &= \frac{26f(G, \mathcal{P}) - 9}{2}, \end{aligned}$$

and hence  $f(G, \mathcal{P}) \leq (2n + 9)/26$ .

**Case 2:**  $t_{\mathcal{P}}(P) \geq 3$ .

By the choice of  $P$ ,  $t_{\mathcal{P}}(P') \geq 3$  for every  $P' \in \mathcal{P}$ . By Lemma 6,  $\sum_{P' \in \mathcal{P}} |X_{\mathcal{P}}(P')| \geq \sum_{P' \in \mathcal{P}} t_{\mathcal{P}}(P')(f(G, \mathcal{P}) - 1) \geq 9(f(G, \mathcal{P}) - 1)$ . This together with Lemma 5 and (2.10) implies that

$$\begin{aligned} n &\geq \frac{3l(G) + \sum_{P' \in \mathcal{P}} |X_{\mathcal{P}}(P')| + 3}{2} \\ &\geq \frac{3(6f(G, \mathcal{P}) - 2) + 9(f(G, \mathcal{P}) - 1) + 3}{2} \\ &= \frac{27f(G, \mathcal{P}) - 12}{2}, \end{aligned}$$

and hence  $f(G, \mathcal{P}) \leq (2n + 12)/27$ .

This completes the proof of Theorem 4.  $\square$

To conclude this section, we propose the following conjecture.

**Conjecture 8** *Let  $G$  be a connected graph, and let  $\mathcal{P} \subseteq \mathcal{L}(G)$  with  $|\mathcal{P}| = 3$ . If there exists a path  $P \in \mathcal{P}$  with  $t_{\mathcal{P}}(P) = 2$ , then  $f(G, \mathcal{P}) = 0$ .*

If Conjecture 8 is true, then we can improve the upper bound of  $f(G, \mathcal{P})$  in Theorem 4 to  $(2n + 12)/27$  (by the argument in the proof of Theorem 4).

### 3 Bounding the value of $f(G, \mathcal{P})$ by a sublinear function

A function  $g$  is *sublinear* if  $\lim_{n \rightarrow +\infty} \frac{g(n)}{n} = 0$ . It follows from the definition that, if  $g$  is sublinear, then for any two constants  $c_0, c_1$ , we have  $g(c_0t + c_1) < t$  for any large  $t$ . Here we pose the following new conjecture, which concerns Conjecture 2. Although Conjecture 9 is seemingly weaker than Conjecture 2, we will show that Conjecture 9 is indeed equivalent with Conjecture 2.

**Conjecture 9** *There exists a sublinear non-decreasing function  $g$  such that for every connected graph  $G$  of order  $n$  and every subset  $\mathcal{P}$  of  $\mathcal{L}(G)$  with  $|\mathcal{P}| = 3$ ,  $f(G, \mathcal{P}) \leq g(n)$ .*



To prove that this seemingly weaker conjecture is equivalent to Conjecture 2, we first show that for a given graph  $G$  with a set  $\{P_1, P_2, P_3\}$  of three longest paths one can choose a subdivision of  $G$  so that subdivisions of  $P_i$ 's  $i = 1, 2, 3$  are the new longest paths and show that the minimum distance from these three subdivided paths in the subdivided graph grows linearly in the order of subdivision. For the exact statement we introduce the following notation.

Let  $G$  be a connected graph and let  $\mathcal{P} = \{P_1, P_2, P_3\}$  be a set of three longest paths. Let  $G'$  be obtained by adding a new edge to each end-vertex of  $P_i$ 's,  $i = 1, 2, 3$ ; thus, minimum of two and maximum of six new vertices and edges are added. Let  $P'_i$ ,  $i = 1, 2, 3$  be the path corresponding to  $P_i$  with two new edges at the two ends. We define  $G^t$  to be the graph obtained from  $G'$  by subdividing each edge  $t$  times. Let  $P_i^t$ ,  $i = 1, 2, 3$  be the corresponding path of  $P'_i$  in  $G^t$ . We write  $\mathcal{P}^t = \{P_1^t, P_2^t, P_3^t\}$ . Also, let  $V_{f(G, \mathcal{P})} = \{v \in V(G) \mid \sum_{P \in \mathcal{P}} d_G(v, V(P)) = f(G, \mathcal{P})\}$ .

We have the following proposition.

**Proposition 10** *Given a connected graph  $G$  and a set  $\mathcal{P} = \{P_1, P_2, P_3\}$  of three longest paths, the set  $\mathcal{P}^t = \{P_1^t, P_2^t, P_3^t\}$  is a set of three longest paths of  $G^t$ . Furthermore,  $f(G^t, \mathcal{P}^t) = (t + 1)f(G, \mathcal{P})$ .*

*Proof.* The first assertion is easy to check. To prove the second assertion, we show that a vertex of  $V_{f(G^t, \mathcal{P}^t)}$  could be chosen as an original vertex of  $G$ . The assertion then would follow, as the vertex of  $G$  attaining the distance sum  $f(G, \mathcal{P})$  of  $\mathcal{P}$  satisfies  $(t + 1)f(G, \mathcal{P})$  for the distance sum of  $\mathcal{P}^t$  in  $G^t$  as well.

Now let  $u$  be a vertex attaining the distance sum  $f(G^t, \mathcal{P}^t)$  from  $\mathcal{P}^t$ . It is easy to check that  $u$  is not an end-vertex of  $P_i^t$  for any  $i$ . If  $u \in V(G)$ , then we have nothing to prove. Otherwise  $u$  is a new vertex subdividing an edge, say  $xy$ , of  $G$ . If all the shortest paths from  $u$  to  $P_i^t$ ,  $i = 1, 2, 3$ , go through  $x$  (or  $y$ ) then replacing  $u$  by  $x$  (or  $y$ ) provides a smaller distance sum than  $f(G^t, \mathcal{P}^t)$ , a contradiction. Thus we may assume, without loss of generality, that two of the shortest paths from  $u$  to  $P_i^t$  go through  $x$  and the third one goes through  $y$ . In such a case again by replacing  $u$  by  $x$  we will have a smaller distance sum than  $f(G^t, \mathcal{P}^t)$ , a contradiction. We note that if  $u$  belongs to one or two of these paths then so are  $x$  and  $y$ , thus this would not affect the argument. The contradiction proves that  $u$  must be a vertex of  $G$  and we have  $f(G^t, \mathcal{P}^t) = (t + 1)f(G, \mathcal{P})$ .  $\square$

Keeping the above proposition in mind, we can prove the following theorem.

**Theorem 11** *Conjecture 2 is true if and only if Conjecture 9 is true.*

*Proof.* The “only if” part is trivial, and hence we only show the “if” part.

Suppose that  $G$  together with  $\mathcal{P} = \{P_1, P_2, P_3\}$  is a counterexample for Conjecture 2, i.e.,  $f(G, \mathcal{P}) \geq 1$ . The subgraph of  $G$  induced by edges and vertices of  $P_1, P_2, P_3$  is also a counterexample (where  $\mathcal{P}$  is also a set of non-intersecting three longest paths). Note that, in view of Proposition 3, such a subgraph is connected. Thus we may assume from the start that vertices and edges of  $G$  are union of vertices and edges of  $P_1, P_2, P_3$ . Let  $n_0$  be the number of vertices of  $G$ . Since  $G$  is

union of three paths each of length at most  $n_0 - 1$ , we conclude that  $G$  has at most  $3(n_0 - 1)(< 3(n_0 + 1))$  edges.

Hence, by the construction of  $G^t$ , we have  $|V(G^t)| \leq n_0 + 3(n_0 + 1)t + 6$ . On the other hand, we have  $f(G^t, \mathcal{P}^t) = (t + 1)f(G, \mathcal{P}) \geq t$ . Hence for constants  $c_0 = 3n_0 + 3$  and  $c_1 = n_0 + 6$  we have  $g(c_0t + c_1) \geq t$  (because  $g$  is non-decreasing). However, this contradicts the fact that  $g$  is a sublinear function.  $\square$

In conclusion, Theorem 11 tells us that giving a substantial improvement on the magnitude of the upper bound of  $f(G, \mathcal{P})$  in Theorem 4 settles the longstanding conjecture on intersecting three longest paths in a connected graph.

## Acknowledgments

We would like to thank Dr. Valentin Borozan for a fruitful discussion concerning this paper. The first author would like to thank the laboratory LRI of the University Paris South and Digiteo foundation for their generous hospitality. He was able to carry out part of this research during his visit there. Also, the first author's research is supported by the Japan Society for the Promotion of Science Grant-in-Aid for Young Scientists (B) (20740095). The second author's research is in part supported by the Japan Society for the Promotion of Science Grant-in-Aid for Young Scientists (B) (26800086). The fourth author's research is in part supported by the Japan Society for the Promotion of Science Grant-in-Aid for Young Scientists (B) (25871053), and by Grant for Basic Science Research Projects from The Sumitomo Foundation.

## References

- [1] M. Axenovich, When do three longest paths have a common vertex? *Discrete Math. Algorithms Appl.* 1 (2009) 115-120.
- [2] P. Balister, E. Györi, J. Lehel, R. Schelp, Longest paths in circular arc graphs, *Combin. Probab. Comput.* 13 (2004) 311-317.
- [3] S. F. de Rezende, C. G. Fernandes, D. M. Martin, Y. Wakabayashi, Intersecting longest paths, *Discrete Math.* 313 (2013) 1401-1408.
- [4] P. Erdős, G. Katona (Eds.), *Theory of Graphs, Proceedings of the Colloquium Held at Tihany, Hungary, 1966*, Academic Press, New York, 1968, Problem 4 (T. Gallai), p. 362.
- [5] J. Harris, J. Hirst, M. Mossinghoff, *Combinatorics and Graph Theory*, in: *Undergraduate Texts in Mathematics*, Springer, 2008.
- [6] S. Kravžar, M. Petkovšek, Graphs with nonempty intersection of longest paths, *Ars Combin.* 29 (1990) 43-52.
- [7] C. Thomassen, Planer and infinite hypohamiltonian and hypotraceable graphs, *Discrete Math.* 14 (1976) 377-389.

- [8] H. Walther, Über die Nichtexistenz eines Knotenpunktes, durch den alle längsten Wege eines Graphen gehen, J. Combin. Theory 6 (1969) 1-6.
- [9] H. Walther, H. J. Voss, Über Kreise in Graphen, VEB Deutscher Verlag der Wissenschaften, 1974.
- [10] T. Zamfirescu, On longest paths and circuits in graphs, Math. Scand. 38 (1976) 211-239.
- [11] T. Zamfirescu, Intersecting longest paths or cycles: a short survey, An. Univ. Craiova Ser. Mat. Inform. 28 (2001) 1-9.